Tutorial 1.27

Recall that given a function f on $[-\pi, \pi]$ the Fourier Series of f is defined by

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

where $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \, dy.$

Verify $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \, dy$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny \, dy$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny \, dy, \quad n \geq 1$$

Pf: Clearly, $\hat{f}(0) = a_0$

When $n \geq 1$,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) e^{-iny} dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) (\cos ny - i \sin ny)\, dy$$

$$= \frac{1}{2}(a_n - ib_n)$$

$$\hat{f}(-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) e^{iny} dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) (\cos ny + i \sin ny)\, dy$$

$$= \frac{1}{2}(a_n + ib_n)$$

Thus, $\hat{f}(n) + \hat{f}(-n) = a_n$

$\hat{f}(n) - \hat{f}(-n) = -ib_n$.

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} = a_0 + \sum_{n=1}^{\infty} (\hat{f}(n) e^{inx} + \hat{f}(-n) e^{-inx})$$

$$= a_0 + \sum_{n=1}^{\infty} \Big[(\hat{f}(n) + \hat{f}(-n) \cos nx) + (\hat{f}(n) - \hat{f}(-n)) i \sin nx \Big]$$

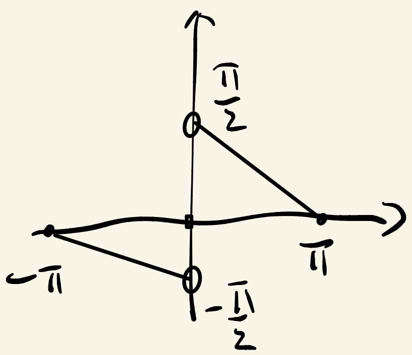
$$= a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + (-ib_n) i \sin nx]$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$\square$

Define $f : [-\pi, \pi] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & \text{if } -\pi \le x < 0, \\ 0, & \text{if } x = 0 \\ \frac{\pi}{2} - \frac{x}{2}, & \text{if } 0 < x \le \pi. \end{cases}$$



We will compute the Fourier Series of f and discuss its convergence property.

By Q1, $f \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Note that f is an odd function.

Then $\quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, dy = 0$

When $n \ge 1$, $a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \cos ny\, dy = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny\,dy$$

$$\begin{pmatrix} f(y) \text{ is odd} \\ \sin ny \text{ is odd} \\ \Rightarrow f(y)\sin ny \text{ is even} \end{pmatrix}$$

$$= \frac{2}{\pi}\int_0^{\pi} f(y)\sin ny\,dy$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{\pi - y}{\cancel{2}}\sin ny\,dy$$

$$= \frac{1}{\pi}\int_0^{\pi} (\pi - y)\sin ny\,dy$$

integral by part ←

$$= \frac{1}{\pi}\left[(\pi - y)\left(-\frac{1}{n}\cos ny\right)\Big|_{y=0}^{\pi} - \int_0^{\pi} -\left(-\frac{1}{n}\cos ny\right)dy\right]$$

$$= \frac{1}{\pi}\left[0 - \left(-\frac{\pi}{n}\right) - \frac{1}{n}\int_0^{\pi}\cos ny\,dy\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{1}{n^2}\sin ny\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\cdot\frac{\pi}{n} = \frac{1}{n}.$$

Hence, the Fourier Series of f is $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$.

Then, we will discuss the convergence property of $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$.

Q: Does $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converge pointwisely on $[-\pi, \pi]$.

A: Yes !

Proof: When $x=0$, it is trivial.

When $x \in [-\pi, \pi] \setminus \{0\}$,

$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is the imaginary part of the

complex Series $\sum_{n=0}^{\infty} \frac{e^{inx}}{n}$.

It suffices to show $\sum_{n=0}^{\infty} \frac{e^{inx}}{n}$ is convergent

when $x \in [-\pi, \pi] \setminus \{0\}$.

(Dirichlet's Test)

If $\{a_n\}$ is a sequence of real numbers and

$\{b_n\}$ is a sequence of complex numbers

satifying

- $\{a_n\}$ is monotonic
- $\lim_{n\to\infty} a_n = 0$
- $\left|\sum_{n=1}^{N} b_n\right| \le M$ for every $N \in \mathbb{N}$.

Then $\sum_{n=0}^{\infty} a_n b_n$ is convergent

Fix $x \in [-\pi, \pi] \setminus \{0\}$

Let $a_n = \frac{1}{n}$ and $b_n = e^{inx}$

$$\left|\sum_{n=0}^{N} b_n\right| = \left|\sum_{n=0}^{N} e^{inx}\right| = \left|\frac{1-e^{inx}}{1-e^{ix}}\right| \leq \frac{2}{|1-e^{ix}|}$$

By Dirichlet's Test, $\sum_{n=0}^{\infty} \frac{e^{inx}}{n}$ converges for any $x \in [-\pi, \pi] \setminus \{0\}$.

Thus $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converges for any $x \in [-\pi, \pi]$.

Rmk: Using Dirichlet's Test, you can show

$\forall \delta > 0$, $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converges uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$.

Q: Does $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converge uniformly on $[-\pi, \pi]$?

A: No!

Pf: Recall that

Uniform convergence $\Leftrightarrow$ Uniform Cauchy.

To see this, by Cauchy's Criteria, we need to show $\exists\ \varepsilon_0 > 0$, $(n_k) \to \infty$, $(m_k) \to \infty$ and $(x_k) \in [-\pi, \pi]$ s.t. $\left| \sum_{n=n_k}^{m_k} \frac{\sin n x_k}{n} \right| > \varepsilon_0$.

Let $\varepsilon_0 = \frac{\sqrt{2}}{4}$ $\quad n_k = k$, $m_k = 2k$, $x_k = \frac{\pi}{4k}$

when $\quad n \in [k+1, 2k] = [n_k + 1, m_k]$,

$$\sin n x_k = \sin \frac{n\pi}{4k} \in \left[\frac{1}{\sqrt{2}}, 1\right].$$

Thus
$$\sum_{n=n_k+1}^{m_k} \frac{\sin n x_k}{n} = \sum_{n=n_k+1}^{m_k} \frac{\sin \frac{n\pi}{4k}}{n}$$

$$> \sum_{n=n_k+1}^{m_k} \frac{\frac{1}{\sqrt{2}}}{2k}$$

$$= k \cdot \frac{\frac{1}{\sqrt{2}}}{2k} = \frac{\sqrt{2}}{4} = \varepsilon_0 .$$

By Cauchy's Criteria, we show that

$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is not uniformly convergent on $[-\pi, \pi]$.

$\square$